NORTH-HOLLAND

Uniqueness of Strongly Regular Graphs Having Minimal *p*-Rank

René Peeters*
Department of Econometrics
Tilburg University
P.O. Box 90153
5000 LE Tilburg
The Netherlands

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ABSTRACT

Let Γ be a strongly regular graph with adjacency matrix A. Let I be the identity matrix and let p be a prime. We study the p-rank of the matrices $A + cI$ for integral c and want to characterize among all strongly regular graphs with a given parameter set those for which a p-rank is minimal.

1. INTRODUCTION

As usual (see e.g. [10], [11] or [24]), a strongly regular graph with parameters (v, k, λ, μ) is a graph Γ with v vertices not complete or null, in which the number of common neighbors of x and y is k, λ, or μ according as x and y are equal, adjacent or nonadjacent, respectively. The complement

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of a strongly regular graph is again strongly regular with parameters

$$(v, v-k-1, v-2k+\mu-2, v-2k+\lambda).$$

Of course, not every parameter set is feasible, but the problem of determining all those parameter sets for which a strongly regular graph really exists if far from being solved. On the other hand, there are parameter set for which many (nonisomorphic) graphs are known. For instance, there are at least 105 strongly regular graphs with parameter set (36, 14, 4, 6) (see [24]). For some parameter sets it has been proved that there exists a unique strongly regular graph with these parameters (see for instance [7]). Among these there are two infinite families:

The *Triangular Graphs* $T(n)$ are the line graphs of K_n for $n \geqslant 3$ and are strongly regular with parameter set $(\frac{1}{2}n(n-1), 2(n-2), n-2, 4)$. If $n \neq 8$, then $T(n)$ is the unique strongly regular graph with these parameters; if $n = 8$, there are three more strongly regular graphs with these parameters. These are called the Chang graphs.

The *Lattice Graphs* $L_2(n)$ are the line graphs of $K_{n,n}$ for $n \geqslant 2$ and are strongly regular with parameters $(n^2, 2(n-1), n-2, 2)$. If $n \neq 4$, then $L_2(n)$ is the unique strongly regular graph with these parameters; if $n = 4$, there is one more strongly regular graph with these parameters: the Shrikhande graph.

All strongly regular graphs on fewer than 25 vertices are uniquely determined by their parameters, except for the parameter set (16, 6, 2, 2) (and its complementary set) referred to above. Furthermore, uniqueness has only been shown for the parameter sets of Table I (cf. [7, 6]).

TABLE 1
STRONGLY REGULAR GRAPHS THAT ARE UNIQUELY DETERMINED BY THEIR PARAMETERS

v	k	λ	μ	Name
27	10	1	5	Schläfli
50	7	0	1	Hoffman-Singleton
56	10	0	2	Gewirtz
77	16	0	4	co sub HiS (M_{22})
81	20	1	6	sub $GQ(3,9)$
100	22	0	6	Higman-Sims
112	30	2	10	$GQ(3,9)$ (sub McL)
162	56	10	24	co sub McL
275	112	30	56	McLaughlin

For a strongly regular graph Γ with parameters (v, k, λ, μ), let A be its adjacency matrix. Then, apart from the valency k, A has two more eigenvalues, r and s say, with multiplicities f and g, satisfying

$$\lambda - \mu = r + s, \qquad \mu - k = rs,$$

$$f + g = v - 1, \qquad k + fr + gs = 0.$$

If $f = g$ (the so-called "half case"), we have that

$$(v, k, \lambda, \mu) = (4t + 1, 2t, t - 1, t)$$

for some $t \geqslant 1$ and Γ has the same parameters as its complement and eigenvalues $(-1 \pm \sqrt{v})/2$. Otherwise, the eigenvalues r and s are integers.

Let p be a prime number and $c \in \mathbb{Z}$; then it can be proven (see [5]) that the p-rank of the matrix $A + cI$ is completely determined by the parameters of Γ except possibly for the p-ranks

$$r_p\left(A + \frac{p+1}{2} I\right) \quad \text{with } p \mid v \text{ in the half case, and}$$

$$r_p(A - sI) \quad \text{with } p \mid r - s \text{ if } r \text{ and } s \text{ are integral}$$

for which we only have the upper bound $\min\{f + 1, g + 1\}$. So only for these p-ranks (which will be referred to as the relevant p-ranks of Γ, or just the p-rank of Γ if one specific value of p is meant) the structure of Γ can play a role, and so these p-ranks can be used to distinguish between nonisomorphic strongly regular graphs with the same parameters. Our aim in this paper is the characterize strongly regular graphs by their parameters and (one of) their relevant p-ranks, especially those which have among all strongly regular graphs with a given parameter set the smallest relevant p-rank for some prime number p. In other words, we want to prove the uniqueness of strongly regular graphs given their parameter set and the minimality (among all strongly regular graphs with this parameter set) of some relevant p-rank.

Results of this type for designs are called rigidity theorems in the book by Assmus and Key [2] and originate from the work of Hamada [17], who conjectured that the designs of points and μ-flats in a t-dimensional projective or affine geometry over some finite field $\mathbb{F}_q$ of characteristic p are uniquely determined by their (design) parameters and the minimality of the p-rank of their incidence matrix. The conjecture was proved to be true for some special cases but turned out to be false in its full generality. Restricted to the designs of points and lines of a projective plane of order q, the conjecture is still open. See [2] for an overview of the results concerning Hamada's conjecture. The (perhaps somewhat vague) idea behind this paper is similar to the idea behind Hamada's conjecture, namely, that (for a suitable prime p) the "nicest" or "most regular" strongly regular graph with some given parameters can be characterized by its parameters and the minimality of its p-rank, or at least that strongly regular graphs for which the p-rank attains the minimal value must be "nice" in some sense.

EXAMPLE 1.1. As mentioned earlier, there are four nonisomorphic strongly regular graphs with parameters (28, 12, 6, 4) and spectrum $12^1, 4^7, -2^{20}$. It can be found in [5] (and it will follow later on) that $r_2(A) = 6$ for $T(8)$ and $r_2(A) = 8$ for the three Chang graphs. So $T(8)$ is uniquely determined by its parameters and the minimality of the 2-rank.

EXAMPLE 1.2. Both $L_2(4)$ and the Shrikhande graph with parameters (16, 6, 2, 2) and spectrum $6^1, 2^6, -2^9$ have $r_2(A) = r_2(A + J) = 6$, so we cannot distinguish between these two graphs by their relevant ranks.

2. PRELIMINARIES

2.1. *Some Linear Algebra*

We denote the row space of a matrix A by $\langle A \rangle$ or by $\langle A \rangle_p$ if it is desirable to mention explicitly that the field is $\mathbb{F}_p$. Vectors will be row vectors and $\mathbf{1}$ and $\mathbf{0}$ denote the all-one vector and the zero vector, respectively. We denote the all-one matrix by J and the all-zero matrix by O. Sometimes we will call $\langle A \rangle_p$ the linear code over $\mathbb{F}_p$ generated by A.

A code C is called *self-orthogonal* if $C \subset C^{\perp}$. The following easy results are well known.

LEMMA 2.1 (cf. [27]). *If C is a binary self-orthogonal code and each row of the generator matrix of C has weight divisible by 4, then so does every code word.*

LEMMA 2.2 (cf. [5]). *The 2-rank of a symmetric integral matrix with zero diagonal is even.*

LEMMA 2.3. *Let A be a symmetric integral matrix will all diagonal elements equal to* 1; *then* $\mathbf{1} \in \langle A \rangle_2$.

Proof. Consider A as a matrix over $\mathbb{F}_2$ and let $r := r_2(A)$. Then there exists a matrix $B \in \mathbb{F}_2^{v \times r}$ such that $A = BB^T$. Since B has full column rank, there exists a vector $\underline{x} \in \mathbb{F}_2^v$ such that $\underline{x}B = \mathbf{1}$. Let $\underline{b} \in \mathbb{F}_2^r$ be a row of B; then $\underline{bb}^T = 1$, so also $\mathbf{1}\underline{b}^T = 1$ and hence $\underline{x}A = \underline{x}BB^T = \mathbf{1}B^T = \mathbf{1}$. ■

LEMMA 2.4 (cf. [5]). *Let A be an integral matrix of order v with constant column sums m; then we have,* $\mathbf{1} \in \langle A \rangle_p$ *if* $p \nmid m$, *and we have* $\mathbf{1} \notin \langle A \rangle_p$ *if* $p \mid m$ *and* $p \nmid v$.

LEMMA 2.5. *Let A be an integral matrix of order v such that* $AA^T = O \pmod p$. *If* $p \nmid v$, *then* $\mathbf{1} \notin \langle A \rangle_p$.

From Smith normal form it follows that (see [5]):

LEMMA 2.6. *Let M be a nonsingular integral matrix of order v and suppose* $p^k \parallel \det(M)$. *Then* $r_p(M) \geqslant v - k$.

2.2. *Switching*

Let $\Gamma = (V, E)$ be a graph and let $\{V_1, V \setminus V_1\}$ be a partition of V; then (as in [24]) we define the result of switching Γ with respect to this partition to be the graph $\Gamma' = (V, E')$ whose edges are those edges of Γ contained in V_1 or $V \setminus V_1$ together with the pairs $\{v_1, v_2\}$ with $v_1 \in V_1$, $v_2 \in V \setminus V_1$ for which $\{v_1, v_2\} \notin E$. The graphs Γ and Γ' are said to be "switching-equivalent."

Suppose Γ is strongly regular with parameters satisfying $v + 4rs + 2r + 2s = 0$. If Γ' is regular, then Γ' is again strongly regular, either with different parameters, or with the same parameters but nonisomorphic to Γ. or isomorphic to Γ. If $V_1 = \Gamma(x)$, that is the set of neighbors of x in Γ, for an arbitrary vertex x of Γ, then Γ' is the disjoint union of x and a strongly regular graph with $k = 2\mu$.

From [5] we mention the following results for the p-rank of switching-equivalent graphs. If Γ and Δ are switching-equivalent graphs with adjacency matrices A and B, respectively, then for odd p we have

$$r_p\left(A - \frac{p+1}{2}J + cI\right) = r_p\left(B - \frac{p+1}{2}J + cI\right) \quad \text{for all } c \in \mathbb{Z}. \quad (1)$$

If $p = 2$ and Δ is obtained from Γ by switching with respect to a set with characteristic vector $\underline{\chi}$, then

$$\langle A + b_1 J + cI \rangle + \langle \mathbf{1}, \underline{\chi} \rangle$$
$$= \langle B + b_2 J + cI \rangle + \langle \mathbf{1}, \underline{\chi} \rangle \quad \text{for all } b_1, b_2, c \in \{0.1\}.$$

If, furthermore, Δ has an isolated vertex, x say, we have $\underline{\chi} \in \langle A \rangle$ and $\mathbf{1} \notin \langle B \rangle$, so either

1. $r_2(A) = r_2(B) + 2$, and hence $\mathbf{1} \in \langle A \rangle$ and $\underline{\chi} \notin \langle B \rangle$ or
2. $r_2(A) = r_2(B)$, with $\mathbf{1} \notin \langle A \rangle$ and $\underline{\chi} \in \langle B \rangle$.

EXAMPLE 2.1. $T(8)$ and the three Chang graphs are switching-equivalent, so if A and B are the adjacency matrices of any two of these four graphs, then by (1), $r_3(A + J + 2I) = r_3(B + J + 2I)$ and by Lemma 4 we have $\mathbf{1} \notin \langle A + J + 2I \rangle_3$, $\mathbf{1} \in \langle A + 2I \rangle_3$ and $\mathbf{1} \in \langle A + 2J + 2I \rangle_3$. It is easy to check that if A is the adjacency matrix of $T(8)$, then $r_3(A + 2I) = 8$, so for all four graphs we have that $r_3(A + 2I) = r_3(A + J + 2I) + 1 = r_3(A + 2J + 2I) = 8$.

3. PALEY GRAPHS

Let q be a prime power, $q \equiv 1 \pmod 4$, and let $P(q)$ be the graph with vertex set $\mathbb{F}_q$ where two vertices are adjacent whenever their difference is a nonzero square. Then $P(q)$ is called the Paley graph of order q. The graph $P(q)$ is strongly regular with parameters

$$\left(q, \frac{q-1}{2}, \frac{q-1}{4} - 1, \frac{q-1}{4}\right)$$

and spectrum

$$\left(\frac{q-1}{2}\right)^1, \left(\frac{-1+\sqrt{q}}{2}\right)^{(q-1)/2}, \left(\frac{-1-\sqrt{q}}{2}\right)^{(q-1)/2}.$$

Clearly a graph with these parameters has the same parameters as its complement, and in fact $P(q)$ and $\overline{P(q)}$ are isomorphic.

It is proved in [5] that if A is the adjacency matrix of $P(q)$ with $q = p^e$, then

$$r_p\left(A + \frac{p+1}{2} I\right) = \left(\frac{p+1}{2}\right)^e,$$

so in particular the p-rank of $P(p)$ is equal to $(p+1)/2$. This in fact holds for all strongly regular graphs with the same parameters as $P(p)$ and is a consequence of the following theorem which reduces the relevant p-ranks for the half case to those p for which $p^2 \mid v$.

THEOREM 3.1. *Let Γ be a strongly regular graph with parameters $(4t+1, 2t, t-1, t)$ for some t and let A be its adjacency matrix. If $p \parallel v$, then*

$$r_p\left(A + \frac{p+1}{2} I\right) = 2t + 1.$$

Proof. Since $(A + [(p+1)/2]I)^2 \equiv tJ \pmod{p}$, clearly

$$r_p\left(A + \frac{p+1}{2} I\right) \leqslant 2t + 1.$$

Let the spectrum of Γ be $2t^1, r^{2t}, s^{2t}$ with $rs = -t$ and $r + s = -1$. Define

$$A' := \left(\begin{array}{c|c} \dfrac{v+p}{2} & \mathbf{1} \\ \hline \mathbf{1}^T & A + \dfrac{p+1}{2} I \end{array}\right).$$

Now modulo p the sum of the rows of $A + [(p+1)/2]I$ corresponding to the neighbors of some vertex x of Γ minus the sum of the rows corresponding to the nonneighbors of x is equal to $-\frac{1}{2}\mathbf{1}$, so $r_p(A + [(p+1)/2]I) = r_p(A')$. The matrix A' has spectrum

$$\left(\frac{v+p}{2} + \sqrt{v}\right)^1, \left(\frac{v+p}{2} - \sqrt{v}\right)^1, \left(r + \frac{p+1}{2}\right)^{2t}, \left(s + \frac{p+1}{2}\right)^{2t}$$

and hence $p^{2t+1} \parallel \det(A')$, so $r_p(A') \geqslant (v+1)/2 = 2t + 1$. ∎

COROLLARY 3.1. *Let A be the adjacency matrix of a strongly regular graph with the same parameters as a Paley graph of prime order p; then*

$$r_p\left(A + \frac{p+1}{2} I\right) = \frac{p+1}{2}.$$

So the strongly regular graphs with the same parameters as some Paley graph of prime order p cannot be distinguished by their p-rank. The Paley graphs of order 5, 13 and 17 are the unique strongly regular graphs with these parameters, but it can be found in [8] that there exist at least 41 strongly regular graphs with the same parameters as $P(29)$,[1] at least 82 strongly regular graphs cospectral with $P(37)$ (arising from 11 two-graphs) and at least 120 strongly regular graphs cospectral with $P(41)$ (arising from 18 two-graphs). The following example shows that we may expect a more positive result for strongly regular graphs with the same parameters as $P(q)$ for $q = p^e$ with $e > 1$.

EXAMPLE 3.1. (For the results used in this example we refer to [20], in which all nonisomorphic strongly regular graphs with parameter sets $(25, 12, 5, 6)$ and $(26, 15, 8, 9)$ are determined, although at that moment it was not yet known that these were all. This was done by exhaustive computer search (see [26] who refers to [1]).) A latin square of order 5 defines a strongly regular graph with parameters $(25, 12, 5, 6)$ and spectrum $12^1, 2^{12}, -3^{12}$ in the following way: take as vertices the 25 cells of the latin square, two vertices being adjacent whenever they are in the same row or column, or have the same symbol. Up to isomorphism there are two latin squares of order 5; one is isomorphic to the cyclic latin square (whose graph is isomorphic to the Paley graph $P(25)$) and one is not. Let A_1' and A_2', respectively, be the adjacency matrices of these two graphs and, for later use, let A_1 and A_2 be the adjacency matrices of the graphs Γ_1 and Γ_2 consisting of the latin square graph plus an isolated vertex. It is proved in [22] (cf. Theorem 1) that $r_5(A_1' + 3I) = 9$ and $r_5(A_2' + 3I) = 11$.

There are two nonisomorphic Steiner triple systems on 13 symbols. Let N_3 and N_4 be their 26×13 triple-symbol incidence matrices; then $A_3 := N_3 N_3^T - 3I$ and $A_4 := N_4 N_4^T - 3I$ are adjacency matrices of two (nonisomorphic) strongly regular graphs with parameters $(26, 15, 8, 9)$ and spectrum

[1] They arise from 6 nonisomorphic conference two-graphs on 30 vertices and were previously found after incomplete backtracking search in [1]. Independently, E. Spence and F. C. Bussemaker (personal communication) did an exhaustive search and found no more such graphs.

$15^1, 2^{12}, -3^{13}$. According to [17, Table 6.1, No. 12], $r_5(N_3) = r_5(N_4) = 13$, so $r_5(A_i + 3I) = r_5(N_i N_i^T) = 13$ for these two graphs.

There are 15 nonisomorphic strongly regular graphs with parameters (25, 12, 5, 6) and 10 nonisomorphic strongly regular graphs with parameters (26, 15, 8, 9). Let $\mathscr{K}$ be the class of 25 graphs of 26 vertices consisting of the 10 nonisomorphic strongly regular graphs with parameters (26, 15, 8, 9) and the 15 nonisomorphic graphs consisting of an isolated vertex and a strongly regular graph with parameters (25, 12, 5, 6). Then $\mathscr{K}$ falls apart into four equivalence classes under the switching-equivalence relation, each containing one of Γ_i $(i = 1, 2, 3, 4)$. Let $\mathscr{K}_i$ be the class containing Γ_i. Let A and B be the adjacency matrices of two graphs from the same class; then by (1), $r_5(A + 2J + 3I) = r_5(B + 2J + 3I)$. Now, by Lemma 4 if A corresponds to a strongly regular graph on 26 vertices and by Lemma 5 in the other case, we get that $\mathbf{1} \notin \langle A + 2J + 3I \rangle_5$ and $\mathbf{1} \in \langle A + bJ + 3I \rangle_5$ for $b = 0, 1, 2, 3, 4$. So for every $b \in \{0, 1, 2, 3, 4\}$, $r_5(A + bJ + 3I)$ is equal for all graphs in the same class $\mathscr{K}_i$. Since $r_5(A_i + 3I) = 10, 12, 13, 13$ for $i = 1, 2, 3, 4$, respectively, and $\mathscr{K}_1$ contains only Γ_1 and one strongly regular graph, Δ say, with parameters (26, 15, 8, 9), we get that the following graphs are unique given their parameters and the minimality of their 5-rank.

1. $P(25)$ with parameters (25, 12, 5, 6) and $r_5(A + bJ + 3I) = 9$ for $b = 0, 1, 2, 3, 4$.

2. Δ with parameters (26, 15, 8, 9) and $r_5(A + 2J + 3I) + 1 = r_5(A + bJ + 3I) = 10$ for $b = 0, 1, 3, 4$.

3. $\overline{\Delta}$ with parameters (26, 10, 3, 4) and $r_5(A + 2J + 3I) + 1 = r_5(A + bJ + 3I) = 10$ for $b = 0, 1, 3, 4$.

4. LATIN SQUARE GRAPHS

Let L be a latin square of order n. Its latin square graph Γ is the graph with the cells of L as vertex set, two vertices being adjacent if the two corresponding cells are in the same row or column or have the same symbol. Γ is strongly regular with parameters $(n^2, 3(n-1), n, 6)$ and spectrum $3(n-1)^1, (n-3)^{3(n-1)}, -3^{(n-1)(n-2)}$, so its relevant p-ranks are $r_p(A + 3I)$ with $p \mid n$. A strongly regular graph with the same parameter set as a latin square graph is called a pseudo latin square graph. A pseudo latin square graph is called *geometric* if it is the graph of a latin square. By a theorem of Bose [4] (see also [10]) we have that a pseudo latin square graph on n^2 vertices is geometric if $n > 23$. As in [12], we define the *direct product* of two latin squares L_1 and L_2 of order m and n, respectively, as the latin square of order mn obtained by replacing each entry of L_1 by a copy of L_2. The replacement of entries of L_1 need not be uniform and we can in fact use different latin squares of order n. A latin square that can be obtained in this way from L_1 is called a *nonuniform product* of L_1 and latin squares of order n.

If N is the line-point incidence matrix of a 3-net of order n (latin square of order n) with adjacency matrix A, then $A + 3I = N^T N$. Using the results and terminology of Moorhouse [19], who determines the p-ranks of N for p dividing n, the p-ranks of $N^T N$ are determined in [22] where the following theorem can be found:

THEOREM 4.1. *Let L be a latin square of order n, Γ its latin square graph with adjacency matrix A, and let G be the loop corresponding to L (L is isomorphic to the multiplication table of G); then for $p \mid n$:*

$$\begin{aligned} r_p(A + 3I) &= 3n - 5 && \text{if } p = 2, 2 \parallel n \text{ and } \dim \mathrm{Hom}(G, \mathbb{F}_p) = 1,\\ &= 3n - 6 && \text{if } p = 2, 4 \parallel n \text{ and } \dim \mathrm{Hom}(G, \mathbb{F}_p) = 2,\\ &= 3n - 4 - 2\dim \mathrm{Hom}(G, \mathbb{F}_p) && \text{otherwise.} \end{aligned}$$

Here $\mathrm{Hom}(G, \mathbb{F}_p)$ is the vector space over $\mathbb{F}_p$ of the p-characters of G, that is, the vector space of all maps $\theta : G \to \mathbb{F}_p$ with $\theta(g * h) = \theta(g) + \theta(h)$ for all $g, h \in G$ ($*$ is the binary operation of G). If $\dim \mathrm{Hom}(G, \mathbb{F}_p) = d$, then d is the maximal number such that L is isomorphic to a latin square that is a nonuniform product of the multiplication table of the direct product of d cyclic groups of order p and latin squares of order n/p^d. So if $p^e \parallel n$, then $d \leqslant e$ (cf. [19] or [22]).

Adding up modulo p the rows of $A + 3I + cJ$ (for some $c \in \mathbb{Z}$) corresponding to n cells of L with the same symbol yields $2\mathbf{1}$. So $\mathbf{1} \in \langle A + 3I + cJ \rangle_p$ for all $c \in \mathbb{Z}$ if $p \neq 2$, and by Lemma 3 we have $\mathbf{1} \in \langle A + 3I \rangle_2$. If $2 \mid n$, then by Theorem 1, $r_2(A + 3I)$ is even unless $2 \parallel n$ and $\dim \mathrm{Hom}(G, \mathbb{F}_2) = 1$. Since by Lemma 2, $r_2(A + 3I + J)$ is even,

$$\mathbf{1} \in \langle A + 3I + J \rangle_2$$

unless $2 \parallel n$ and L is a nonuniform product of the multiplication table of C_2 (the cyclic group of two elements) and latin squares of order $n/2$. In that case, $\mathbf{1} \notin \langle A + 3I + J \rangle_2$ and $r_2(A + 3I + J) = r_2(A + 3I) - 1 = 3n - 6$. In all other cases, $r_p(A + 3I + cJ)$ with $p \mid n$ is the same for all $c \in \mathbb{Z}$.

Let, for $n > 23$, Γ be the (pseudo) latin square graph on n^2 vertices with adjacency matrix A and let p be a prime dividing n. Let $e \geqslant 1$ be the integer such that $p^e \parallel n$. By the foregoing we get the following results:

1. If $p = 2$ and $e = 1$, then $r_2(A + 3I) \in \{3n - 4, 3n - 5\}$ and the minimum is attained iff Γ corresponds to a latin square that is a nonuniform product of the multiplication table of C_2 and latin squares of order $n/2$.
2. If $p = 2$ and $e = 2$, then $r_2(A + 3I) \in \{3n - 4, 3n - 6\}$ and the minimum is attained iff Γ corresponds to a latin square that is a nonuniform product of the multiplication table of C_2 and latin squares of order $n/2$.
3. If either $p = 2$ and $e \geqslant 3$ or $p \neq 2$, then

$$r_p(A + 3I) \in \{3n - 4, 3n - 6, 3n - 8, \ldots, 3n - 4 - 2e\}$$

and the minimum is attained iff Γ corresponds to a latin square that is a nonuniform product of the multiplication table of the direct product of e copies of the cyclic group of order p and latin squares of order n/p^e.

So, for instance, the latin square graph of the multiplication table of $C_3 \times C_3 \times C_3$ is uniquely determined by its parameters and the minimality of its 3-rank, which is 71, and the latin square graph of the multiplication table of C_{35} is uniquely determined by its parameters and the minimality of both its 5-rank and its 7-rank, which are both 99. All latin square graphs corresponding to a latin square that is a nonuniform product of the multiplication table of C_{14} and latin squares of order 2 have minimal 7-rank equal to 78 as well as minimal 2-rank equal to 78.

5. SYMPLECTIC GRAPHS

Let V be a vector space of dimension $2n$ over $\mathbb{F}_2$ provided with a nondegenerate symplectic form $B : V \times V \to \mathbb{F}_2$. Now the symplectic graph $\mathscr{S}p(2n, 2)$ is the graph of the perpendicular relation induced on the nonzero vectors of V. So by definition its complementary graph $\overline{\mathscr{S}p(2n, 2)}$ has adjacency matrix

$$A = \left[B(u, v) \right]_{u, v \in V \setminus \{0\}}$$

of 2-rank $2n$. There are essentially two quadratic forms $Q : V \to \mathbb{F}_2$ which have B as their associated symplectic form: one, Q^+ say, with $2^{2n-1} + 2^{n-1}$ zeros and one, Q^- say, with $2^{2n-1} - 2^{n-1}$ zeros. For each of these there is a partition of $\mathcal{S}p(2n, 2)$ into two subgraphs $\mathcal{N}^{\epsilon}_{2n}$ and $\mathcal{S}^{\epsilon}_{2n}$ of vectors achieving value 1 or 0 under Q^{ϵ}, respectively. So for the adjacency matrix of $\overline{\mathcal{N}^{\epsilon}_{2n}}$ we have

$$A = \left[B(u, v)\right]_{u, v \in V\setminus\{0\},\, Q^{\epsilon}(u) = Q^{\epsilon}(v) = 1},$$

and for the adjacency matrix of $\overline{\mathcal{S}^{\epsilon}_{2n}}$:

$$A = \left[B(u, v)\right]_{u, v \in V\setminus\{0\},\, Q^{\epsilon}(u) = Q^{\epsilon}(v) = 0},$$

so both have 2 rank equal to $2n$. $\mathcal{S}p(2n, 2)$, $\mathcal{N}^{\epsilon}_{2n}$ and $\mathcal{S}^{\epsilon}_{2n}$ are strongly regular with parameters as displayed in Table 2.

A graph G is said to possess the *cotriangle property* if, for every pair $\{x, y\}$ of nonadjacent vertices in G, there exists a third vertex z forming a subgraph $T = \{x, y, z\}$ isomorphic to $\overline{K_3}$ having the property that any vertex u of G not lying in the cotriangle T is adjacent to exactly one or all of the vertices of T. Similarly, a graph G is said to have the *triangle property* if, whenever $\{x, y\}$ is an edge in G, a third vertex z adjacent to both x and y can be found such that any vertex of G not lying in the triangle $\{x, y, z\} = T$ is adjacent to one or all members of T. In terms of the adjacency matrix of its complement, G has the cotriangle property if and only if the sum modulo 2 of two rows of the adjacency matrix of $\overline{G}$ corresponding to two adjacent (in $\overline{G}$) vertices is again a row of this adjacency matrix, and G has the triangle property if and only if the sum modulo 2 of two rows of the adjacency matrix of $\overline{G}$ corresponding to two nonadjacent vertices is again a row of this matrix.

Using this, it follows straightforwardly from the definitions that the graphs $\mathcal{S}p(2n, 2)$ and $\mathcal{N}^{\epsilon}_{2n}$ satisfy the cotriangle property and the graphs $\mathcal{S}p(2n, 2)$ and $\mathcal{S}^{\epsilon}_{2n}$ satisfy the triangle property. There are partial converses to this (see [16] and [25]):

THEOREM 5.1 (Cotriangle Theorem). *If G is a finite graph possessing the cotriangle property such that $\overline{G}$ is connected and no two different vertices of G have the same set of neighbors then either G consists of a single vertex, of G is one of the graphs $\overline{T(n)}$ $(n \geqslant 3)$, $\mathcal{S}p(2n, 2)$ $(n \geqslant 2)$ or $\mathcal{N}^{\epsilon}_{2n}$ $(n \geqslant 3)$.*

THEOREM 5.2 (Triangle Theorem). *Let G be a finite graph possessing the triangle property. Then either*

1. *G contains a vertex x adjacent to every other vertex of G, or*
2. *G is isomorphic to one of the graphs $\overline{K_n}$, $\mathcal{S}p(2n, 2)$, $\mathcal{S}^{\epsilon}_{2n}$.*

It turns out that a strongly regular graph with the same parameters as the symplectic graph or one of the graphs appearing from its partitions with minimal 2-rank must have the triangle property or the cotriangle property, which leads to the following theorem.

THEOREM 5.3. *For every $n \geqslant 2$, the strongly regular graphs $\mathcal{S}p(2n,2)$, $\mathcal{S}_{2n}^{\epsilon}$, $\mathcal{N}_{2n}^{\epsilon}$ and their complements are uniquely determined by their parameters and the minimality of the 2-rank, which is $2n+1$ and $2n$, respectively.*

Proof. Let G be a strongly regular graph with the same parameters as $\mathcal{S}p(2n,2)$. Then all the $2^{2n}-1$ rows of its adjacency matrix A are different, so $r_2(A) \geqslant 2n$ and we have equality if and only if the sum modulo 2 of any two rows of A is again a row of A. So if $r_2(A) = 2n$, then $\overline{G}$ must have the cotriangle property as well as the triangle property so $\overline{G} = \mathcal{S}p(2n,2)$. Since $\mathbf{1} \notin \langle A \rangle_2$ and by Lemma 3 we have $\mathbf{1} \in \langle J + A \rangle_2$, the 2-rank of $\mathcal{S}p(2n,2)$ is $2n+1$. If A' is the adjacency matrix of any strongly regular graph with the same parameters as $\mathcal{S}p(2n,2)$, then $\mathbf{1} \in \langle A' + I \rangle_2$ and $r_2(J + A' + I)$ is even, which implies that also $\mathcal{S}p(2n,2)$ is uniquely determined by its parameters and the minimality of its 2-rank.

Let G be a strongly regular graph with the same parameters (v, k, λ, μ) as $\overline{\mathcal{S}_{2n}^{\epsilon}}$ and adjacency matrix A; then all v row vectors have weight k and the sum modulo 2 of two row vectors has weight $2(k-\lambda)$ or $2(k-\mu) = k$ according to if the two corresponding vertices are adjacent or not. Now the maximal number of pairs of adjacent vertices for which the sum modulo 2 of

TABLE 2
PARAMETERS OF THE STRONGLY REGULAR GRAPHS RELATED TO THE SYMPLECTIC GRAPHS

Name	v	k	λ r	μ s
$\mathcal{S}p(2n,2)$	$2^{2n}-1$	$2^{2n-1}-2$	$2^{2n-2}-3$ $2^{n-1}-1$	$2^{2n-2}-1$ $-2^{n-1}-1$
$\mathcal{N}_{2n}^{+}$	$2^{2n-1}-2^{n-1}$	$2^{2n-2}-1$	$2^{2n-3}-2$ $2^{n-2}-1$	$2^{2n-3}+2^{n-2}$ $-2^{n-1}-1$
$\mathcal{N}_{2n}^{-}$	$2^{2n-1}+2^{n-1}$	$2^{2n-2}-1$	$2^{2n-3}-2$ $2^{n-1}-1$	$2^{2n-3}-2^{n-2}$ $-2^{n-2}-1$
$\mathcal{S}_{2n}^{+}$	$2^{2n-1}+2^{n-1}-1$	$2^{2n-2}+2^{n-1}-2$	$2^{2n-3}+2^{n-1}-3$ $2^{n-1}-1$	$2^{2n-3}+2^{n-2}-1$ $-2^{n-2}-1$
$\mathcal{S}_{2n}^{-}$	$2^{2n-1}-2^{n-1}-1$	$2^{2n-2}-2^{n-1}-2$	$2^{2n-3}-2^{n-1}-3$ $2^{n-2}-1$	$2^{2n-3}-2^{n-2}-1$ $-2^{n-1}-1$

the two corresponding rows is the same vector is $k - \lambda$. Since the total number of edges is $\frac{1}{2}vk$, $\langle A \rangle_2$ contains at least $vk/[2(k - \lambda)]$ different vectors of weight $2(k - \lambda)$ and also at least v vectors of weight k and the null-vector. Now it turns out that for the chosen parameters

$$\frac{vk}{2(k-\lambda)} + v + 1 = 2^{2n},$$

so $r_2(A) \geqslant 2n$ and equality implies that the sum modulo 2 of two rows of A corresponding to two nonadjacent vertices must again be a row of A, so $\overline{G}$ has the triangle property. By the Triangle Theorem, $\overline{G}$ is isomorphic to $\mathscr{S}_{2n}^{\epsilon}$ for which indeed $r_2(A) = 2n$. By the foregoing $\mathbf{1} \notin \langle A \rangle_2$ and by Lemma 3 we have $\mathbf{1} \in \langle A \rangle_2$, so that 2-rank of $\mathscr{S}_{2n}^{\epsilon}$ is $2n + 1$, and since $\mathbf{1} \in \langle A' + I \rangle_2$ for the adjacency matrix A' of any strongly regular graph with the same parameters as $\mathscr{S}_{2n}^{\epsilon}$ and $r_2(A' + I + J)$ is even, also $\mathscr{S}_{2n}^{\epsilon}$ is uniquely determined by its parameters and the minimality of the 2-rank.

Finally, the case $\mathscr{N}_{2n}^{\epsilon}$ can be proved similarly to the case $\mathscr{S}_{2n}^{\epsilon}$. ■

REMARK 5.1. In fact, the Cotriangle Theorem and the Triangle Theorem are far too heavy machinery to prove the uniqueness of the symplectic graph $\mathscr{S}p(2n, 2)$ and its complement by their parameters and their minimum 2-rank. If A is the adjacency matrix of a strongly regular graph, Γ say, with the same parameters as the complement of $\mathscr{S}p(2n, 2)$ having 2-rank $2n$, then there exists a $2^{2n} - 1 \times 2n$-matrix B such that

$$A = B \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n\right) B^T.$$

Now the rows of B must be precisely all $2^{2n} - 1$ nonzero vectors from $\mathbb{F}_2^{2n}$, so Γ is isomorphic to the complement of $\mathscr{S}p(2n, 2)$.

REMARK 5.2. If A is the adjacency matrix of a strongly regular graph with the same parameters as $\mathscr{S}p(2n, 2)$, then $A + I$ can be regarded as the point-block incidence matrix of a BIB design with parameters

$$(v, b, r, k, \lambda) = (2^{2n} - 1, 2^{2n} - 1, 2^{2n-1} - 1, 2^{2n-1} - 1, 2^{2n-2} - 1).$$

It has been proved in [18] that among all BIB designs with parameters $(2^{t+1} - 1, 2^{t+1} - 1, 2^t - 1, 2^t - 1, 2^{t-1} - 1)$, the incidence matrix of the

BIB design defined by the points and $(t-1)$-flats in the finite projective geometry $PG(t,2)$ has the minimum 2-rank, which is equal to $t+2$. It is easy to see that for $t+1=2n$, this BIB design is isomorphic to the BIB design defined by $A+I$, where A is the adjacency matrix of $\mathscr{S}p(2n,2)$.

REMARK 5.3. Up to isomorphism there are 80 Steiner triple systems on 15 symbols (cf. [9]). Let N be the 35×15-triple-symbol incidence matrix of such a Steiner triple system; then $A := NN^T - 3I$ is the adjacency matrix of a strongly regular graph, Γ say, with parameters $(35,18,9,9)$. If $r_2(N) = 15-d$, then $r_2(NN^T) = r_2(A+3I) = 15-2d$, since if $N\underline{x}^T = \mathbf{0}$, for some $\underline{x} \in \mathbb{F}_2^{15}$, then $\underline{x}$ must have weight 8 and $\underline{x}\underline{y}^T = 0$ for each $\underline{y} \in \mathbb{F}_2^{15}$ with $N\underline{y}^T = \mathbf{0}$. So by Theorem 3 we have $r_2(N) \geqslant 11$. If N is the incidence matrix of the Steiner triple system defined by the points and the lines in the projective geometry $PG(3,2)$, then by [17, Table 7.1] we have $r_2(N) = 11$, so $r_2(NN^T) = 7$ and hence Γ is isomorphic to $\mathscr{S}_6^+$. The 80 nonisomorphic Steiner triple systems give rise to nonisomorphic strongly regular graphs, so the Steiner triple system defined by the points and the lines of $PG(3,2)$ is uniquely determined by its parameters and the minimality of the 2-rank of its incidence matrix. It is easy to see that $r_3(N) = 14$ and $r_3(NN^T) = 13$ for all Steiner triple systems on 15 symbols.

REMARK 5.4. $\mathscr{S}_6^-$ is the Schläfli Graph, which by Theorem 3 is uniquely determined by its parameters and the minimality of its relevant 2-rank, which is 7. But it follows from the multiplicities of the eigenvalues that 7 is an upper bound for this 2-rank, so the Schläfli graph is uniquely determined by its parameters alone.

REMARK 5.5. The complement of $\mathscr{N}_6^+$ is isomorphic to $T(8)$ with 2-rank equal to 6. From the multiplicities of the eigenvalues it follows that a graph with these parameters has 2-rank at most 8; using Lemma 2 we get that the three Chang graphs as well as their complements must have 2-rank equal to 8.

REMARK 5.6. According to [24], there are at least 16,448 nonisomorphic strongly regular graphs with the same parameters as $\mathscr{N}_6^-$ and there are at least 1853 nonisomorphic strongly regular graphs with the same parameters as $\mathscr{S}_6^+$.

Using Theorem 3, we can prove the uniqueness of a strongly regular graph with given parameter set having minimal 2-rank in one more case.

$G_2(2)$ has a rank 3 representation on 36 points, giving rise to a strongly regular graph (which we will call $G_2(2)$ as well) will parameters (36, 14, 4, 6) and spectrum $14^1, 2^{21}, -4^{14}$, which is a subconstituent of the Hall-Janko graph.

THEOREM 5.4. *The strongly regular graph $G_2(2)$ and its complement are uniquely determined by their parameters and the minimality of their 2-rank, which is 8 in both cases.*

Proof. By Theorem 3, the complement of $\mathcal{S}_6^+$ is the unique strongly regular graph with parameters (35, 16, 6, 8) and (minimal) 2-rank 6. According to [24], there is a unique possibility of switching this graph plus an isolated vertex, x say, into a regular graph with valency 14. This yields the graph $G_2(2)$. Let A and B be the adjacency matrices of $G_2(2)$ and $\overline{\mathcal{S}_6^+} \cup \{x\}$, respectively. Let $\underline{\chi}$ be the characteristic vector of the set of neighbors of x in $G_2(2)$; then

$$\langle A \rangle_2 + \langle \mathbf{1}, \underline{\chi} \rangle_2 = \langle B \rangle_2 + \langle \mathbf{1}, \underline{\chi} \rangle_2 .$$

Clearly $\mathbf{1} \notin \langle B \rangle_2$ and $\underline{\chi} \in \langle A \rangle_2$. Since for $\overline{\mathcal{S}_6^+}$ the parameters k, λ and μ are all even, the code generated by the rows of B is self-orthogonal. Furthermore, k is divisible by 4, so by Lemma 1 all words in the code have weight divisible by 4. So $\underline{\chi} \notin \langle B \rangle$ and hence $\mathbf{1} \in \langle A \rangle_2$ and $r_2(A) = r_2(B) + 2 = 8$. Since $\mathbf{1} \in \langle A + J \rangle_2$, also $r_2(A + J) = 8$.

Finally, let Γ be any strongly regular graph with parameters (36, 14, 4, 6) with adjacency matrix A and let B be the adjacency matrix of the strongly regular graph, Δ say, with parameter set (35, 16, 6, 8) that appears from Γ after switching with respect to $\Gamma(x)$ for some vertex x of Γ. Then by the same arguments as above,

$$r_2(A) = r_2(A + J) = r_2(B) + 2.$$

So $r_2(A) = r_2(A + J) \leqslant 8$ implies that $\Delta = \overline{\mathcal{S}_6^+}$ and hence $\Gamma = G_2(2)$. ∎

REMARK 5.7. The complement of $\mathcal{S}_6^+$ plus an isolated vertex can also be switched regular into a strongly regular graph having parameters set (36, 20, 10, 12). According to [24], there are three nonisomorphic strongly regular graphs obtained in this way, one of which if $\overline{\mathcal{N}_6^-}$, so the other two graphs as well as their complements have 2-rank equal to 8.

6. GENERALIZED QUADRANGLES WITH $s = 3$

If GQ is a generalized quadrangle of order (s, t) (cf. [21], notation: $GQ(s,t)$), then the collinearity graph of GQ is strongly regular with parameters $((s + 1)(st + 1), s(t + 1), s - 1, t + 1)$. The relevant p-ranks of a strongly regular graph with these parameters are $r_p(A + (t + 1)I)$ for $p \mid s + t$. In this part we will only consider strongly regular graphs with the same parameters as a collinearity graph of a $GQ(s, t)$ with $s = 3$. We use the same notation as in [21], in which also most results used in this part can be found.

If $s = 1$, the only feasible values for t are 1, 3, 5, 6 and 9. A $GQ(3, 1)$ is just the 4×4-grid which is already considered in the introduction. The collinearity graph of the (unique) $GQ(3, 9)$ is one of the strongly regular graphs that is already uniquely determined by its parameters. There exists no $GQ(3, 6)$, and by Haemers ([14]) there exists no strongly regular graph with the same parameters as the collinearity graph of a $GQ(3, 6)$.

There are precisely two GQ's of order $(3, 3)$, namely, $W(3)$ and its dual $Q(4, 3)$, and there is precisely one $GQ(3, 5)$, namely, $T_2(O)$ arising from a complete oval in $PG(2, 4)$. For the relevant p-ranks of their collinearity graphs we mention the following lemmas. (A *triad* (*of points*) is a triple of pairwise noncollinear points. Given a triad T, a *center* of T is a point that is collinear with all three points of T.)

LEMMA 6.1. *Let A be the adjacency matrix of the collinearity graph of a GQ of order (s, t) with s and t odd. If each triad has an even number of centers, then*

$$r_2(A) = r_2(A + J) = (s - 1)(t + 1) + 2.$$

Proof. Straightforward; see also [3]. ∎

GQ's that satisfy the condition of the lemma are $T_2^*(Q)$, $Q(4, q)$ and those of order (q, q^2) (q odd). Together with the results from [5], we get the following relevant p-ranks for the collinearity graphs of the three mentioned GQ's:

	$W(3)$	$Q(4, 3)$	$T_2^*(O)$
$r_2(A)$	16	10	14
$r_3(A + I)$	11	15	

The collinearity graphs of a $GQ(3,3)$ and a $GQ(3,5)$ have parameters $(40, 12, 2, 4)$ and $(64, 18, 2, 6)$, respectively. So for graphs with these parameters, the neighbor graph of each vertex is regular with degree 2 so it is a disjoint union of cycles. But we can say more.

LEMMA 6.2. *Let G be a strongly regular graph having parameter set* $(40, 12, 2, 4)$ *or* $(64, 18, 2, 6)$; *then for every vertex x, G_x consists of cycles of length divisible by* 3 *and every vertex not adjacent to x is adjacent to precisely $c/3$ vertices of each c-cycle in G_x.*

Proof. See Haemers [13] for graphs with the first parameter set. For graphs with the second parameter set the proof is equivalent. See also the survey by Haemers [15]. ∎

Using this lemma we can characterize $W(3)$ by its 3-rank:

THEOREM 6.1. *The collinearity graphs of the GQ $W(3)$ and its complement are uniquely determined by their parameters and the minimality of their 3-rank, which are* 11 *and* 10, *respectively.*

Proof. Let G be a strongly regular graph with parameters $(40, 12, 2, 4)$ with adjacency matrix A, and let x be a vertex of G. Then by previous lemma, G_x is isomorphic to one of the following graphs: $4C_3$, $2C_3 + C_6$, $C_3 + C_9$, $2C_6$ or C_{12}. If G_x is isomorphic to one of the last four, then using that every vertex not adjacent to x is adjacent to $c/3$ vertices from every c-cycle in G_x, if follows straightforwardly that $r_3(A + I) \geqslant 12$ (see Lemma 1 of the Appendix). So $r_3(A + I) \geqslant 12$ unless, for every vertex x of G, G_x consists of four triangles, which means that G is the collinearity graph of a $GQ(3,3)$. Since $W(3)$ and $Q(4,3)$ are the only GQ's of order $(3,3)$ and their collinearity graphs have 3-rank equal to 11 and 15, respectively, the collinearity graph of $W(3)$ is uniquely determined by its parameters and the minimality of its 3-rank. Using Lemma 4, we get that for the adjacency matrix, $r_3(A + I) = r_3(A + I + J) = r_3(A + I + 2J) + 1 = 11$. ∎

Now let us consider 2-ranks. If G is a strongly regular graph with adjacency matrix A and parameters $(40, 12, 2, 4)$ for which, for some vertex x its neighbor graph G_x is isomorphic to $2C_3 + C_6$, $C_3 + C_9$ or C_{12}, then by Lemma 1 we get that $r_2(A) \geqslant 12$. In case G_x is isomorphic to $4C_3$ or $2C_6$, we only get $r_2(A) \geqslant 10$, so for a strongly regular graph with parameters $(40, 12, 2, 4)$, the minimal value of $r_2(A)$ is 10, and this minimum is only attained by the collinearity graph of $Q(4,3)$ or maybe by a graph that is locally $2C_6$. However, in [23] it is proved that there exists no srg(40, 12, 2, 4) that is locally $2C_6$, so we have the following theorem:

THEOREM 6.2. *The collinearity graphs of the GQ $Q(4,3)$ and its complement are uniquely determined by their parameters and the minimality of their 2-rank, which is* 10 *in both cases.*

Now, let G be a strongly regular graph with parameters (64, 18, 2, 6) and let, for a vertex x, G_x be its neighbor graph. It follows straightforwardly that $r_2(A) \geqslant 14$ if G_x is isomorphic to $6C_3$ or $3C_6$, and by Lemma 1, $r_2(A) \geqslant 16$ in the other nine cases. So the minimal 2-rank of a strongly regular graph with parameters (64, 18, 2, 6) is 14 and the minimum is only attained by the collinearity graph of $T_2^*(O)$ or maybe by a graph that is locally $3C_6$. In [23] it is proved that there are precisely two srg(64, 18, 2, 6) that are locally $3C_6$. Since these two graphs turn out to have 2-ranks 16 and 18, we have the following result.

THEOREM 6.3. *The collinearity graphs of the GQ $T_2^*(O)$ and its complement are uniquely determined by their parameters and the minimality of the 2-rank, which is* 14 *in both cases.*

Finally, Table 3 displays the results of this paper for the smallest parameter sets of strongly regular graphs that do not determine a graph uniquely. The smallest open cases are the 3-ranks of strongly regular graphs with parameter sets (35, 16, 6, 8), (36, 14, 4, 6) and (36, 20, 10, 12).3

APPENDIX

LOWER BOUNDS

In this part we will give lower bounds for the 2- and 3-ranks of a srg(40, 12, 2, 4) and for the 2-rank of a srg(64, 18, 2, 6), given the neighbor graph of some vertex of the graph. We denote this neighbor graph by the lengths of its cycles, where the exponents denote the multiplicity of a cycle with a given length.

LEMMA A.1. *Let G be a srg(40, 12, 2, 4) with adjacency matrix A. If G contains a vertex for which the neighbor graph is $3^2 + 6$, $3 + 9$, 6^2 or* 12,

TABLE 3

SMALLEST PARAMETER SETS OF STRONGLY REGULAR GRAPHS THAT DO NOT DETERMINE A GRAPH UNIQUELY

v	k	λ	μ	r^f	s^g	p	diag	u.b.	min	attained by
16	6	2	2	2^6	$(-2)^9$	2	0	6	6	$L_2(4)$ + Shrikhande
25	12	5	6	2^{12}	$(-3)^{12}$	5	3	12	9	$P(25)$
26	10	3	4	2^{13}	$(-3)^{12}$	5	3	13	10	switched $P(25)$
28	12	6	4	4^7	$(-2)^{20}$	2	0	8	6	$T(8) = \mathrm{co}\mathscr{N}_6^+$
						3	2	8	8	all
29	14	6	7			29	15	15	15	all
35	16	6	8	2^{20}	$(-4)^{14}$	2	0	14	6	$\mathrm{co}\,\mathscr{S}_6^+$
						3	1	15	?	???
36	14	4	6	2^{21}	$(-4)^{14}$	2	0	14	8	$G_2(2)$
						3	1	15	?	???
36	20	10	12	2^{20}	$(-4)^{15}$	2	0	16	6	$\mathrm{co}\mathscr{N}_6^-$
						3	1	16	?	???
37	18	8	9			37	19	19	19	all
40	12	2	4	2^{24}	$(-4)^{15}$	2	0	16	10	$Q(4, 3)$
						3	1	16	11	$W(3)$
41	20	9	10			41	21	21	21	all
⋮										
63	32	16	16	4^{27}	$(-4)^{35}$	2	0	28	6	$\mathrm{co}\,\mathscr{S}p(6, 2)$
64	18	2	6	2^{45}	$(-6)^{18}$	2	0	18	14	$GQ(3, 5)$

then

$$r_3(A + I) \geqslant 12.$$

If G contains a vertex for which the neighbor graph is $3^2 + 6$, $3 + 9$ *or* 12, *then*

$$r_2(A) \geqslant 12.$$

Let G be a $srg(64, 18, 2, 6)$ *with adjacency matrix A. If G contains a vertex for which the neighbor graph is* $3^4 + 6$, $3^3 + 9$, $3^2 + 6^2$, $3^2 + 12$, $3 + 6 + 9$, $3 + 15$, $6 + 12$, 9^2 *or* 18 *then*

$$r^2(A) \geqslant 16.$$

Proof. Let $P = \{P_1, P_2, P_3\}$ and $Q = \{Q_1, Q_2, Q_3\}$ be two partitions of the vertex set of G such that there are no edges with one endpoint in P_1 and one in Q_2 and also no edges between P_2 and Q_1. If we partition the rows of A (or $A + I$) according to P and the columns according to Q, then we get the following partition of the matrix A:

$$A = \begin{pmatrix} A_{11} & O & A_{13} \\ O & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

and clearly $r(A) \geqslant r(A_{13}) + r(A_{31}) + r(A_{22}) - r(A_{11})$. This argument solves all the cases of Lemma 1, except if G is a srg(40, 12, 2, 4) and the neighbor graph of some vertex is the 12-cycle.

Let us first consider the 3-rank of $A + I$ for a srg(40, 12, 2, 4). Let x be a vertex of G. If the neighbor graph G_x of x is $3^2 + 6$ or $3 + 9$, apply the above observation to the partitions consisting of $P_1 = Q_1$: the union of all triangles in G_x; $P_2 = Q_2$: the remaining neighbors of x, and $P_3 = Q_3$: the remaining vertices. In case $G_x = 3^2 + 6$, we then find $r_3(A + I) \geqslant 5 + 5 + 4 - 2 = 12$ and in the case $G_x = 3 + 9$, we find $r_3(A + I) \geqslant 3 + 3 + 7 - 1 = 12$. In the case $G_x = 6^2$, apply the observation to the partitions consisting of P_1: a 6-cycle of G_x; Q_2: the other 6-cycle of G_x; $P_2 = Q_1 = \emptyset$, and P_2 and Q_3: the vertices of G not contained in P_1 and Q_1, respectively. We then find $r_3(A + I) \geqslant 6 + 6 = 12$.

Now consider the 2-rank of the adjacency matrix of a srg(40, 12, 2, 4) G. By Lemma 2 this rank is even. Let x be a vertex from G. If $G_x = 3^2 + 6$, then apply the observation to the partitions consisting of P_1: the three vertices of a triangle of G_x; Q_2: the other nine vertices of G_x; $P_2 = Q_1 = \emptyset$, and P_3 and Q_3: the vertices of G not in P_1 and Q_1, respectively. We then find $r_2(A) \geqslant 3 + 8 = 11$, so $r_2(A) \geqslant 12$. If $G_x = 3 + 9$, take for P_1 the vertices of the 9-cycle from G_x; for Q_2 the vertices of the triangle from G_x, and take for P_3 and Q_3 the vertices not in P_1 and Q_2, respectively. So $P_2 = Q_1 = \emptyset$. We find again $r_2(A) \geqslant 3 + 9 = 12$.

If G is a srg(64, 18, 2, 6) which contains a vertex for which the neighbor graph is one of the nine cases mentioned in the lemma, then similarly to above partitions, P and Q can be found yielding that $r_2(A) \geqslant 16$.

Finally, let G be a srg(40, 12, 2, 4) with a vertex, x say, for which the neighbor graph is a 12-cycle. Let A' be the adjacency matrix of the 12-cycle; then $r_3(A' + I) = r_2(A') = 10$. However, if we add up modulo 3 the rows (columns) of A corresponding to the 12-cycle, we do not get the zero-vector since each vertex not adjacent to x is adjacent to four vertices of the 12-cycle, so $r_3(A + I) \geqslant 12$. Partition the 12-cycle into two cocliques, C_1 and C_2 say, of size 6 and suppose $r_2(A) = 10$. Then the sum modulo 2 of the rows of A corresponding to the vertices of C_i $(i = 1, 2)$ is the zero-vector, so the four common neighbors of x and a vertex not adjacent to x lie all in one of the two C_i's or there are two in each. By counting triples (c_1, c_2, x') such that $c_1 \in C_1$, $c_2 \in C_2$ and x' is adjacent to c_1 and c_2, but not to x, we find that 21 of the 27 non-neighbors of x have two neighbors in each C_i, three have four neighbors in C_1 and none in C_2, and three have four neighbors in C_2 and none in C_1. Consider a row of A corresponding to a non-neighbor, x' say, of x that has four neighbors in C_2 and none in C_1. If $r_2(A) = 10$, then this row is a linear combination (modulo 2) of rows of A corresponding to $C_1 \cup C_2$. Since x' has no neighbors in C_1, its corresponding row is a linear combination (mod 2) of rows of A corresponding to C_1 only. All vertices in C_1 are adjacent to x, so the row of x' is the sum (mod 2) of an even number of rows of C_1. The sum (mod 2) of all rows of C_1 is the zero-vector, so without loss of generality the row of x' is the sum (mod 2) of two rows of C_1. But the sum (mod 2) of two rows of C_1 has weight 16, a contradiction. So we have $r_2(A) \geqslant 12$.

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